Maximum Entropy Production and the Earth's Climate

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SUMMARY

The steady-state climate model based on a constraint of maximum entropy production (MEP) developed by Paltridge in the mid seventies is updated and examined. The original maximization routines for calculating the climate distribution are replaced by the semi-analytic technique of O'Brien and Stephens. This technique incorporates maximization of convective (sensible and latent) heat flux *HLE* from surface to atmosphere, which was a second and more-or-less independent constraint built into the original model and which was later called the convection hypothesis. Here the technique is compared with a modification in which the convection hypothesis is replaced by the maximization of the actual entropy production associated with *HLE*. In this modified form the overall model finds the climate distribution that maximizes the entropy production associated with all the turbulent (i.e. non-radiative) horizontal and vertical energy transport of the planet. With some slight tuning, the modification gives results much the same as with the convection hypothesis, but introduces the need to specify the partition of horizontal energy flux between ocean and atmosphere. Various problems with the original model are identified and resolved, among them being an unrealistic definition of atmospheric temperature which at the time led to calculated meridional energy fluxes about half those observed.

KEYWORDS: maximum entropy production MEP climate modelling

1. INTRODUCTION

Paltridge (1975 and 1978) found that the global distributions of average cloud cover and surface temperature could be simulated remarkably well using only two broad thermodynamic constraints together with the requirement for energy balance both at the top of the atmosphere (TOA) and at the earth's surface. The first constraint concerned maximization of the overall rate of flow of entropy to space. It was calculated as the sum over all the model grid boxes i of the net radiative outputs R_{Ni} from the TOA, each divided by an atmospheric or planetary temperature T_{ai} defined in terms of the outward long-wave radiative flux from each box. From steady state considerations, the maximization of entropy flow defined in this way is equivalent to maximization of the rate of entropy production associated with the horizontal (basically meridional) energy flows within the system. The second of the constraints concerned maximization of the vertical flux *HLE* of sensible and latent heat from ground to atmosphere in each grid box. This second constraint was later called the 'convection hypothesis'. Mathematically, the two constraints can be expressed as the maximization of:

$$\dot{S}_p = \sum_i \Delta X_i / T_{ai} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1a)$$

and of

$$HLE_i \quad \forall \quad i \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1b)$$

where $\dot{S}_p$ is the global rate of production of entropy associated with horizontal energy flow and ΔX_i is the net convergence of horizontal energy flow into or out of an individual grid box *i*.

Over the years since then, the principle of maximum entropy production (MEP) has been argued and developed from many points of view and applied successfully in a number of contexts not necessarily related to the Earth's climate. See the review by Osawa et al. (2003). Development was restricted mainly because there was no solid physical argument supporting the existence of such a principle. Then Dewar (2003 and 2005) derived a statistical-mechanical proof of the concept which, while still the subject of argument in a number of forums, has led to greatly increased interest in the subject. The latest developments in the light of Dewar's analysis are discussed in a recent book edited by Kleidon and Lorenz (2005) devoted to the role of MEP in irreversible thermodynamics, and in an article by Whitfield (2005).

Because of the renewed interest, it is appropriate to examine Paltridge's original model in an attempt to resolve a number of its inconsistencies and to update its physics. There is for instance the question of the relation of the second constraint (maximization of *HLE*) to the MEP principle of the first, and whether the two constraints can be handled within the one MEP umbrella. There is a problem with the definition of atmospheric temperature T_a, and there is a related problem with matching the meridional energy fluxes of the 1978 version of the model with those of the 1975 version and with observation. (The meridional fluxes of the 1978 version of the model are about half the observed fluxes). The analytic techniques developed by O'Brien and Stephens (1995) need to be incorporated both for clarity and for the increased computing efficiency they provide. And at the purely mechanical level, the values of some of the input parameters need to be updated. In particular the solar input as a function of latitude needs to take account of the tilt of the Earth's axis.

2. BACKGROUND TO THE PHYSICS

The MEP principle is effectively a statement that any system capable of a large number of steady states will adopt that particular steady state which maximizes its rate of entropy production. The requirement for a large number (or indeed a continuous spectrum) of potential steady states implies that the transport of energy through the system must be a turbulent process so that there can be sufficient degrees of freedom for the transport coefficients to adopt the required value.

The energy flows in the earth-atmosphere system are driven by radiation fluxes which, while not exactly linear functions of their sources and sinks, are certainly not turbulent. Therefore the entropy production associated with the conversion of radiation to thermal energy need not be considered as relevant to the MEP principle other than as a source of the energy that enters the system at the temperature of the boundary of the turbulent medium. This concept is borne out by the work of O'Brien (1997), who showed that the maximum in entropy production associated with the Earth's meridional energy flow appears as a very small perturbation on the much larger total production. The total production is dominated by the downgrading of solar to thermal energy, and that component does not have a maximum.

Figure 1 is a representation of a narrow latitude zone $\boldsymbol{i}$ with its atmosphere and ocean/land specifically separated so that reference can be made to the separate atmospheric and oceanic inputs ($X_a+\Delta X_a$ and $X_o+\Delta X_o$) and outputs (X_a and X_o) of meridional turbulent energy flow. The corresponding boundary temperatures are $T_a+\Delta T_a$, $T_o+\Delta T_o$ and T_a, T_o. ΔX_a and ΔX_o are the meridional energy convergences into atmosphere and ocean of the latitude zone, and their sum is equal to the total convergence ΔX appearing in eq. (1a). The short-wave (S) and long-wave (L) radiation streams R absorbed or emitted by atmosphere and ocean are subscripted appropriately as shown in the figure. HLE is the turbulent transfer of sensible (H) and latent (LE) heat to the atmosphere from ocean or ground.

The total rate of export ('exchange') of entropy out of the atmospheric box of the figure is given by

$$\dot{S}_{ae} = \frac{X_a}{T_a} - \frac{X_a + \Delta X_a}{T_a + \Delta T_a} + \frac{R_{LT} - R_{ST} + R_{SO} - R_{LO} - HLE}{T_a} \qquad \ldots\ldots (2)$$

With a little manipulation, and using the requirement for steady-state energy balance that the numerator of the third term is equal to ΔX_a, eq. (2) reduces to the positive quantity

$$\begin{aligned} \dot{S}_{ae} &= -X_a \, \Delta(1/T_a) \\ &= \dot{S}_{ap} \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$\dot{S}_{ap}$ is the rate of internal production of entropy associated with the downgrading of energy as it flows down the meridional temperature gradient of the atmospheric 'box'. It is equal to $\dot{S}_{ae}$ since the system is assumed to be in steady state and the internal production must equal the total export. In fact eq. (3) can be written down more-or-less directly from the definition of the entropy production associated with energy flowing down a temperature gradient, as can the equivalent expression for internal oceanic production $\dot{S}_{op}$, and as can also the entropy production associated with the turbulent flux of HLE through the temperature change at the boundary between the ocean and atmosphere boxes. Thus the total turbulent (i.e. non-radiative) entropy production within the latitude zone is given by

$$\dot{S}_p^{turb} = (HLE)\,(1/T_a - 1/T_o) \; - \; X_a \, \Delta(1/T_a) \; - \; X_o \, \Delta(1/T_o) \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Energy balance of the oceanic box ensures that

$$HLE \;=\; R_{SO} - R_{LO} + \Delta X_0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Using eq. (5) to substitute for *HLE* in eq. (4), integrating eq. (4) from pole to pole over all latitude zones (the integration is by parts for the last two terms on the right-hand-side) and recognizing that the polar values of meridional flux are zero, we have for the total turbulent entropy production of the model planet

$$\dot{S}_p^{tot} = \int (R_{SO} - R_{LO})(1/T_a - 1/T_o) + \int (\Delta X_a + \Delta X_o)/T_a \;\ldots\ldots\ldots. \; (6a)$$

$$= \int (R_{SO} - R_{LO})(1/T_a - 1/T_o) - \int (R_{NET}/T_a) \;\ldots\ldots\ldots \; (6b)$$

$$\approx \int (HLE)(1/T_a - 1/T_o) - \int (R_{NET}/T_a) \;\ldots\ldots\ldots\ldots \; (6c)$$

$$= \;\; \dot{S}_{pHLE} \; + \; \dot{S}_p \;\ldots\ldots\ldots\ldots\ldots (6d)$$

where the two entropy production terms on the right of the final eq. (6d) are defined by the respective integrals of the preceding eq. (6c).

Equation (6b) is virtually identical to that derived by Osawa et al. (2003). R_{NET} is the net radiative *input* R_{ST}-R_{LT} to the top-of-the-atmosphere, which by energy balance is equal to the divergence (negative convergence) of meridional energy flow. The 'approximately equal' sign in eq. (6c) concerns the substitution of *HLE* for the R_{S0}-R_{L0} of eq. (6b) rather than the substitution of *HLE*-ΔX_0 as indicated by eq. (5). The neglect of ΔX_o is possible because the integral of ΔX_o over all latitude zones is small relative to the integral of *HLE*. Its divergent (negative) low-latitude values are almost balanced by its convergent (positive) high-latitude values.

The second integral on the right-hand-side of eqs (6b) and (6c) together with its negative sign is numerically equal to the entropy production maximized by Paltridge as the first constraint in his original papers. (To be precise, he *minimized* the integral without its negative sign. He thereby minimized – that is, found the most negative value of – the negative inward flow of entropy, and confused the situation for some years thereafter by referring to the climate system as one of minimum entropy exchange rather than one of maximum entropy production). In any event, the $\dot{S}_p$ of eq. (6d) is exactly that of eq. (1a).

Paltridge's second constraint (the maximization of *HLE*) is obviously related to the first integral on the right-hand-side of eq. (6c), but concerns only the surface-to-atmosphere energy flow rather than its associated entropy production. And in practice it concerns only individual boxes rather than the global integral and is thereby only a local constraint. In the present work, a comparison is made between the use of this original second constraint (case A) and the use of another 'second constraint' (case B) whereby an approximation to the full entropy production factor $(HLE)(1/T_a\text{-}1/T_o)$ of eq. (6c) is maximized for each box.

In either case, the overall operation of the model is as follows: Climate is defined in terms of the surface temperature T_i, cloud cover θ_i, vertical heat flux HLE_i, and the convergence of horizontal energy flow ΔX_i in each of the geographical grid boxes *i*.

Given ΔX for a box, it is possible to calculate θ, T and HLE for that box (and other parameters which depend on them such as atmospheric temperature T_a) by using the energy balances at the TOA and at the surface, together with the second constraint of maximizing either the vertical energy flux HLE (case A) or the entropy production associated with HLE (case B). The model then calculates the particular distribution of ΔX_i that yields a maximum in the global entropy production defined as in eq.(1a).

Referring specifically to case B, application of the second constraint in the calculation of θ, T and HLE of each box is equivalent to maximization of the first integral of eq. (6c) provided that the maximum of that integral corresponds to the sum of the individual maximum entropy productions $(HLE)(1/T_a-1/T_o)$ of each grid box. With this proviso, which is equivalent to assuming the first and second constraints are physically separable, the use of both constraints in the model boils down to maximizing the total entropy production $\dot{S}_p^{tot}$ associated with both the vertical and horizontal turbulent energy transfers of the system.

3. THE UPDATED MODEL

The globe is divided into n_{lat} latitudinal zones so that the surface area of each zone is the same. One-dimensional (1-D) results can be obtained by considering only those n_{lat} latitude zones. Two-dimensional results can be obtained by dividing each zone into a number n_{lon} of equal-area longitude boxes. In the 2-D cases reported here both n_{lat} and n_{lon} are set at 20 so there are 400 boxes in total. Table 1 gives input data pertinent to the ten latitude zones of each hemisphere. The following subsections describe the different elements considered in each box, valid in either a 1-D or 2-D case.

(a) Short-wave radiative parameters

The annual-mean solar radiation input I to the top of the atmosphere of each latitude zone is given in Table 1. The figures are for the effective mid-latitude of each zone and were calculated as the average of seasonal values from Manabe and Moller (1991). These annual means are far more realistic than those in Paltridge's original papers where he assumed that the tilt of the Earth was zero.

The other short-wave input parameters are:

- k fractional short-wave absorption by clear portions of the atmosphere
- k_c fractional short-wave absorption by cloudy portions of the atmosphere
- g_0 albedo of the clear-sky atmosphere on its own
- d_0 albedo of the cloudy-sky atmosphere on its own
- α surface albedo

All of them depend on solar zenith angle, and are therefore in principle a function of latitude. The latitude dependence of k_c is not all that great since most of the short-wave absorption lines are close to saturation for the typical water-vapour paths of a cloudy atmosphere. Therefore it was set to the latitude-independent value quoted in Table 1. The latitude dependence of clear-sky solar absorption k over the oceans is also not very great since evaporation from the water surface maintains relatively high water-vapour optical depths in the atmosphere. On the other hand, k over land may be significantly less than its oceanic value. For the present work it is assumed that its value is reduced in proportion to surface albedo α according to

$$k = k_o - 0.18\,(\alpha - 0.06) \qquad \text{..............(7)}$$

where k_o is the constant oceanic value of k quoted in Table 1. The 0.06 is the albedo of the ocean (see Figure 2), and the 0.18 is a tuned value ensuring that over a desert with an albedo of (say) 0.2, the value of k is reduced by about 10%. The reduction is even greater over the high-albedo regions of polar ice, where again the clear sky atmosphere can be very dry.

The latitude-dependent values of g_0 and d_0 for each zone also appear in Table 1. The sources are quoted in the caption. The surface albedo is a function of geographical position, and the distribution used in the 2-D simulations is shown in Figure 2.

Referring to Figure 3, the overall planetary albedos for both clear and cloudy skies (g_p and d_p respectively) can be calculated to a first approximation from the input data as:

$$g_p = g_0 + \alpha\,(1 - g_0 - k) \qquad \text{................(8a)}$$
$$d_p = d_0 + \alpha\,(1 - d_0 - k_c) \qquad \text{................(8b)}$$

These parameters are used in the energy balance equation at the top-of-the-atmosphere. We also define the corresponding "albedos" for the surface energy balance equation as:

$$g_G = 1 - (1 - \alpha)\,(1 - g_0 - k) \qquad \text{..............(8c)}$$
$$d_G = 1 - (1 - \alpha)\,(1 - g_0 - k_c) \qquad \text{..............(8d)}$$

so that the net short-wave flux into the surface can be calculated directly as $I(1\text{-}g_G)$ for clear skies or $I(1\text{-}d_G)$ for cloudy skies.

(b) Long-wave radiative parameters

The model envisages a two-band long-wave radiating atmosphere. All the atmospheric emission or absorption takes place in only one of the bands which is regarded as 100% opaque at all of its wavelengths. The other band represents the 8 to 14 micron atmospheric window (and the various other smaller windows in the absorption spectrum of the atmosphere) and is regarded as 100% clear at all of its wavelengths. The absorbing gases of the atmosphere are regarded as a blanket, the top of which corresponds to the height (and therefore temperature) from which the radiation within the wavelengths of the absorbing band is emitted upward. The bottom of the blanket radiates downwards from a radiative temperature very close to ground temperature, so that the net exchange of long-wave radiation between ground and atmosphere within the absorbing band is effectively zero.

The emissivity of the atmosphere is determined by the ratio of the width in wavelength terms of the absorbing band relative to that of the total black-body spectrum, but weighted appropriately by the shape of the spectrum. There are two emissivities which might be considered. They are the emissivity ε_a of the clear-sky atmosphere and of the atmosphere below clouds, and the emissivity ε'_a of the atmosphere above the cloud. Transmission of long-wave radiation between ground and space in clear skies (or between ground and cloud base in cloudy skies) can only take place through the atmospheric window whose width is $1-\varepsilon_a$. Transmission between cloud top and space can only take place through the window $1-\varepsilon'_a$. Upward emission of long-wave radiation from the top of the radiating blanket in clear skies is given by $\varepsilon_a.\sigma T_{abt}^4$, and in cloudy skies by $\varepsilon'_a.\sigma T_{abc}^4$, where σ is the Stefan Boltzmann constant and T_{abt} and T_{abc} are the (effective) temperatures at the top of the radiating blankets in clear sky and above cloud respectively.

There are a number of temperatures to be considered. They are:

T	ground or ocean surface temperature (assumed also to be the bulk ocean temperature if there is ocean in the grid box)
T_{abt}	clear sky blanket-top temperature
T_{cb}	cloud base temperature
T_{ct}	cloud top temperature
T_{abc}	cloudy sky above-cloud blanket-top temperature.

In order to keep only one independent temperature per box, we assume that all the temperatures of the above list are proportional to the surface temperature T. This is equivalent to assuming that the vertical thermal structure of the atmosphere is fixed for each box. The parameters introduced to characterize this vertical structure are defined as the constants of proportionality between the potential black-body emission at each temperature and the potential black-body emission of the ground. Thus:

$$\sigma T_{abt}^4 = F^{abt}_G\, \sigma T^4 \qquad \text{..................(9a)}$$
$$\sigma T_{cb}^4 = F^{cb}_G\, \sigma T^4 \qquad \text{..................(9b)}$$
$$\sigma T_{ct}^4 = F^{ct}_{cb}\, F^{cb}_G\, \sigma T^4 \qquad \text{.................. (9c)}$$
$$\sigma T_{abc}^4 = F^{abc}_{ct}\, F^{ct}_{cb}\, F^{cb}_G\, \sigma T^4 \qquad \text{..................(9d)}$$

The values of the constants F are given in Table 1. In the trials of the present model it was decided to set $F^{abc}_{ct}=1$ and $\varepsilon'_a = 0.0$. Either setting effectively removes the radiating atmosphere from above cloud top. This was done to limit the number of parameters that might need to be tuned, and in any event replicates the situation of the original models.

F^{ct}_{cb} (the effective measure of cloud thickness) is set to be latitude dependent to account for the physical fact that cloud top (along with the tropopause) is higher at the more tropical latitudes. F^{abt}_G and F^{cb}_G are assigned single values for the whole Earth. Cloud emissivity ε_c is set at 1.0. Ground emissivity ε varies a little with geography. The polar (over-ice) value is set at 1.0, and everywhere else is set at 0.99 except over the major continental desert regions where it is reduced to 0.9.

Referring to Figure 4, each of the long-wave fluxes relevant to the energy balance at the TOA and at the ground can be described as a fraction of the black-body emission of the ground η (= σT^4) together with parameters defined as follows:

$$
\begin{aligned}
m_a &= \varepsilon_a F^{abt}{}_G \\
m_g &= \varepsilon\,(1 - \varepsilon_a) \\
m_c &= \varepsilon_c\,(1 - \varepsilon'_a)\, F^{ct}{}_{cb}\, F^{cb}{}_G \\
m_{abc} &= \varepsilon'_a\, F^{abc}{}_{ct}\, F^{ct}{}_{cb}\, F^{cb}{}_G \\
n_c &= \varepsilon_c\,(1 - \varepsilon_a)\, F^{cb}{}_G
\end{aligned}
\qquad (10)
$$

(c) The definition of atmospheric temperature T_a

The original versions of the model described by Paltridge in 1975 and 1978 differed from each other in that the calculated latitudinal distribution of meridional energy flux (and hence the flux convergence ΔX) in the 1978 version had values about half of those observed – that is, about half of those calculated from satellite observations of the net radiation fluxes at the top-of-the-atmosphere. This difference was traced during the present work to a change in definition of atmospheric temperature from the first to the second version of the model.

T_a is envisaged as some sort of average of the temperatures at which the various energy fluxes are deposited or exported from the atmosphere of a grid box. In the 1975 version of the model it was defined as $T_a = (R_{LT}/\sigma)^{1/4}$ - that is, in terms of the total upward flux of long-wave radiation at the TOA on the assumption that the spectrum of the radiation can be treated as that of a pure black-body. In the 1978 version, the component of the TOA long-wave flux derived directly from the ground through the atmospheric window in clear skies (i.e. $m_g\eta$) was deleted from the total. This was an attempt to decouple atmospheric and surface temperatures so that variations in T_a and T could in principle be of opposite sign. Suffice it to say that T_a derived in this way was far too low to be representative of any reasonable tropospheric temperature, and as a consequence gave unrealistic values of ΔX.

In the present work T_a is defined as a weighted average of cloud-top temperature and clear-sky top-of-blanket temperature according to:

$$\sigma T_a^{\,4} \;=\; \eta\,\{(1 - \theta)\, F^{abt}{}_G \;+\; \theta\,(m_c + m_{abc})\} \quad \ldots\ldots\ldots\ldots (11)$$

where θ is the fractional cloud cover. This can be expressed in simplified form as

$$\sigma T_a^{\,4} \;=\; \eta\,\{M - N\theta\} \qquad \ldots\ldots\ldots\ldots\ldots(11a)$$

with $M = F^{abt}{}_G$ and $N = F^{abt}{}_G - m_c - m_{abc}$. The definition preserves the decoupling of surface and atmospheric temperature, and ensures that T_a is at least roughly in an acceptable range. In the actual model it is possible to tune T_a by varying the effective value of $F^{abt}{}_G$ with a multiplying factor z_0. For case A (maximization of *HLE* in each box) the tuned value of z_0 is 1.07. For case B (maximization of $HLE(1/T_a - 1/T)$ in each box), the tuned value is 1.09.

(d) Energy balance equations

The energy balance at the top-of-the-atmosphere (TOA) requires for each grid box that:

$$Ly[1-(1-\theta)g_p-\theta d_p]-\eta[(1-\theta)(m_g+m_a)+\theta(m_c+m_{abc})] + \Delta X_a+\Delta X_o = 0 \qquad \text{.......(12)}$$

where L is the solar constant (1368 Wm^{-2}), and y is the ratio of the annual-average projected surface area of the grid box as seen from the Sun to the real surface area. Given the zonal annual-average solar inputs I quoted in Table 1, $y = I/L$. As before, ΔX_a and ΔX_o are the net horizontal energy convergences into the atmosphere and ocean of the box. Their sum is the total convergence ΔX.

Energy balance at the box ground/ocean surface requires that:

$$Ly[1-(1-\theta)g_G-\theta d_G]-\eta[m_g-n_c\theta]-HLE+\Delta X_o = 0 \qquad \text{......(13)}$$

Using the notation of O'Brien and Stephens (1995), these two equations can be written in the form:

$$L(A-B\theta)-\eta(C-D\theta)+\Delta X = 0 \qquad \text{............. (14)}$$
$$L(P-Q\theta)-\eta(R-S\theta)-HLE+\Delta X_o = 0 \qquad \text{.........(15)}$$

where:

$$\begin{aligned} A &= y(1-g_p)\\ B &= y(d_p-g_p)\\ C &= m_g+m_a\\ D &= m_g+m_a-m_c-m_{abc}\\ P &= y(1-g_G)\\ Q &= y(d_G-g_G)\\ R &= m_g\\ S &= n_c \end{aligned} \qquad (16)$$

4. THE SOLVING METHOD

(a) Maximization of HLE and HLE($1/T_a$-$1/T$)

O'Brien and Stephens (1995) provided analytic solutions to the problem of finding the cloud cover θ_{max} and surface temperature T_{max} corresponding to maximum HLE of a grid box when given a particular value of convergence ΔX. These solutions are:

$$\eta_{max}\ (=\sigma T_{max}^4) = L\{B-H(\gamma)^{1/2}\}/D \qquad \text{...........(17)}$$

$$\theta_{max} = \{C-H^{-1}\gamma^{1/2}\}/D \qquad \text{...........(18)}$$

where $H = \{(BS-DQ)/(CS-DR)\}^{1/2}$ and $\gamma = BC-AD-D\Delta X/L$

Having obtained the values of T_{max} and θ_{max} for each box for a particular distribution of ΔX_i, it is possible to calculate the radiation fluxes used to define the corresponding atmospheric temperatures T_{ai} from eq. (11). Then, for the given distribution of ΔX_i, it is possible to calculate the global entropy production $\dot{S}_p$ from eqs (1a) or (6d).

The 'case A' solutions of eqs (17) and (18) are compared in the present work with corresponding solutions for a 'case B' – that is, for the case of maximizing the entropy production $\dot{S}_{pHLE} = HLE(1/T_a\text{-}1/T)$. In order to simplify the mathematics we assume as earlier that the vertical structure of the atmosphere is fixed, so that $T_a = bT$ where b is a constant of proportionality. Since $(1/T_a\text{-}1/T) = (T\text{-}T_a)/TT_a$, it follows that

$$\dot{S}_{pHLE} \propto \frac{HLE}{T} \qquad \ldots\ldots\ldots\ldots (19)$$

and it is this quantity, rather than the full $HLE(1/T_a\text{-}1/T)$, which is maximized in each box for any given distribution of convergence ΔX.

Case B analytic solutions for η_{max} and θ_{max} analogous to those of eqs (17) and (18) are derived in Appendix A as eqs (A4) and (A5). Note that case B requires for the oceanic (i.e non-continental) boxes a specification of ΔX_o as well as ΔX. In the present work it is simply assumed that the ratio $\Delta X_o/\Delta X$ is 0.5. This is a rough approximation to the observed situation, at least as far as zonal averages are concerned, and in any event it turns out (see later) that the sensitivity of θ and T to the ratio is extremely small.

(b) Maximization of the entropy production $\dot{S}_p$

The final step is the maximization of the entropy production associated with the horizontal energy fluxes – that is, to find the distribution of ΔX which yields the maximum of

$$\dot{S}_p = \sum_i \frac{\Delta X_i}{T_{ai}} \qquad \ldots\ldots\ldots\ldots..(20)$$

where the subscript i denotes a box number. Again following O'Brien and Stephens (1995) who normalized the fluxes of the energy balance eqs (14) and (15) as fractions of the solar constant, we deal with the normalized quantities $\eta^N{}_{max} = \eta_{max}/L$ and $\zeta = \Delta X/L$. We write eq. (11a) which defines atmospheric temperature in the form

$$T_a = (L/\sigma)^{1/4} f(\zeta) \qquad \ldots\ldots\ldots\ldots.(21)$$

where $f(\zeta) = \{\eta^N{}_{max}(M\text{-}N\theta)\}^{1/4}$. It is a function only of the parameters A, B, C etc and of the normalized convergence. The normalized entropy production corresponding to eq. (20) is defined by

$$\xi \;=\; L^{-3/4}\sigma^{-1/4}\dot{S}_p \;=\; \sum_i \frac{\zeta_i}{f(\zeta_i)} \qquad \text{.........(22)}$$

and is to be maximized with respect to the distribution of ζ_i subject to the constraint that

$$\sum_i \zeta_i = 0 \qquad \text{...........(23)}$$

Letting β denote a Lagrangian multiplier, we can form the Lagrangian cost function

$$\varphi \;=\; \xi - \beta\sum_i \zeta_i \;=\; \sum_i \left(\frac{\zeta_i}{f_i(\zeta_i)} - \beta\zeta_i \right) \qquad \text{.....(24)}$$

The constrained maximization of ξ is equivalent to the unconstrained maximization of φ with respect to ζ_i and β. Finding that maximum can be performed by solving the system of simultaneous equations

$$\frac{\partial\varphi}{\partial(\zeta_i)} = 0 \quad \forall\, i; \quad and \quad \frac{\partial\varphi}{\partial\beta} = 0 \qquad \text{........(25)}$$

That is, with a little manipulation of the appropriate differentiations of eq. (24), by solving the following simultaneous equations for the distribution of normalised convergence ζ_ι over all the boxes i, with the prime indicating differentiation of f_i with respect to ζ_i.

$$\zeta_i f_i' - f_i + \beta f_i^2 = 0 \quad and \quad \sum_i \zeta_i = 0 \qquad \text{........(26)}$$

It is necessary to provide the value of the derivative of f_i with respect to ζ_i for each of the boxes, and this was done here in either of the cases A or B with an analytic expression for the derivative derived from the equation for η_{max} – that is, from eq. (17) or (A4). The solving of the system of eqs (26) is achieved numerically using either Newton-Raphson's method or Broyden's method (Press et al., 1995).

The overall model was coded in double-precision in the Interactive Data Language (IDL) of Research Systems, Inc.

5. RESULTS and DISCUSSION

The meridional profiles of the zonal-average results from the 2-D version of the model are given in Figure 5 (case A) and Figure 6 (case B). The input parameters for the two cases are those of Table 1, together with the surface albedo distribution of Figure 2. The values of z_0 and k_o for the two cases A and B are slightly different because they were roughly tuned to give a reasonable simulation of observed zonal averages (see the dashed curves of the Figures 5 and 6). The 2-D distributions of cloud amount for the two cases are given in Figures 7 and 8. The 2-D distributions of surface temperature and convergence are extremely 'zonal' and are not shown here.

Their longitudinal variation is quite small so that they present little more information than the zonal averages of Figures 5 and 6.

Table 2 gives for each of cases A and B the sensitivities of the global-average values of θ, T and *HLE* to changes in the individual input parameters.

The obvious point from the various comparisons of Figures 5 to 8 is that, once the slight difference between the tuning of the two cases is accepted, there is not a great deal to choose between cases A and B as far as simulation of the observed distributions is concerned. That is, there is nothing in the results themselves which can resolve the question as to what form of the 'second constraint' built into the model (maximization of *HLE* or maximization of its corresponding entropy production) is physically more acceptable.

Case B is aesthetically more attractive, since both of its constraints concern entropy production. And if one accepts the proviso mentioned earlier that the two constraints are essentially separable (because they have different associated time scales for instance) then the overall model can be viewed most simply as process of maximizing the total entropy production of all the turbulent energy transfer processes within the planet. On the other hand case B has a couple of potential difficulties.

The first is that, unlike case A, there is a requirement to specify the ratio of oceanic to atmospheric energy convergence in each box. The sensitivities of the distributions of cloud and surface temperature to this ratio are quite small, as are the equivalent global mean sensitivities quoted in Table 2. Nevertheless they represent a physical dependence which might make it difficult to use an independent relation to calculate the individual atmospheric and oceanic convergences – as was done for instance in Paltridge's original model. Further, the sensitivity of *HLE* to the ratio is large enough to be significant. Compare, for instance, the distributions of *HLE* in Figures 5 and 6, and see the global-mean sensitivity of *HLE* to the ratio as quoted in Table 2.

The second difficulty concerns a purely practical matter of model operation. O'Brien and Stephens (1995) pointed out that the predicted climate of case A in the original version of the model was surprisingly robust for a system involving maximization over so many variables. Ultimately they proved analytically that the solution was unique. The robustness seems to be true also for both cases A and B in the current version of the model, which has a new definition of atmospheric temperature. The solutions are unaffected by wide variation in the initial guesses for the convergence distributions, including setting the convergences to zero at all latitudes. However there are regions of input parameter space where the model 'runs away' to unrealistic solutions. These are regions where the dependence of $\Delta X/T_a$ on ΔX for a particular box has a minimum at a value of ΔX close to the observed value. In other words, these are regions where there are two possible solutions for the entropy production associated with the horizontal convergence of a box. For case A the physically reasonable parameter space is generally well away from this sort of region. This is not true for case B, and indeed the quoted parameter values for case B in Table 1 are right on the limit of operation. Increases in any of $F^{cb}{}_{G}$, $F^{ct}{}_{cb}$ or ε_a push the maximization process into the region of unrealistic solutions – and, usually, into numerical overflow problems.

The MEP constraint on horizontal convergence tends to smooth out any geographical anomalies of surface temperature. Variation in surface albedo (and indeed in any of the input parameters) tends to be reflected in a change in the amount of cloud rather than in temperature. One consequence is that there is relatively little change in temperature within a latitude zone, and the meridional variation of surface temperature is fairly smooth. Another consequence is that the parameterisation of solar absorption by the atmosphere over land (eq. (7) or some equivalent) is necessary to ensure that cloud cover over desert and polar regions is realistic. The solar input to a box is reduced if its albedo is raised, and this tends to lower its surface temperature. The MEP constraint increases the convergence into the box so as to smooth out the temperature difference, which process in turn raises cloud cover because the partial derivative of θ with respect to ΔX is strongly positive. (This derivative can be obtained by re-arranging eq. (14) and differentiating θ with respect to ΔX at constant η or surface temperature. The derivative is equal to the inverse of the negative net cloud forcing discussed below). Suffice it to say that, in the absence of the parameterisation of eq. (7), the cloud cover over the Sahara would be about 0.8, and the cloud cover over the polar regions would be close to 1.0.

The 2-D cloud cover distributions for case A and case B are very similar, although case B has slightly more geographical variability as a consequence of its slight dependence on the ratio of oceanic to atmospheric convergence.

Figure 9 gives for case A the meridional distributions of cloud forcing – that is, of the partial derivatives of the short-wave, long-wave and net radiative fluxes at the TOA with respect to cloud cover. Also shown are those derivatives multiplied by θ so as to give the 'total' forcing due to the existing cloud cover. The shapes of the distributions of total forcing match quite well the corresponding satellite data which can be obtained from a number of sources (e.g. Harrison et al., 1990; Ramanathan et al., 1989). This is largely due to the reasonably good simulation by the model of the distribution of cloud cover. In terms of global averages, the measured short-wave, long-wave and net total forcings are of the order of –50, 30 and –20 Wm^{-2} respectively. The model global-average long-wave forcing is a little less than the measurement by about 10 Wm^{-2} (so that the net is a little greater in absolute magnitude than measurement by that amount – that is, -30 rather than –20 Wm^{-2}) which suggests that the long-wave parameterisations of the model could be tuned slightly differently.

The sensitivity of the predicted climate distribution to surface emissivity is surprisingly small (see Table 2 for the global average values) and, within the range of uncertainty determined by the approximations and the tuning of the other input parameters, there would be little difference to the results if ε were set to 1.0 everywhere.

6. CONCLUSION

The model's simulation of the real Earth's annual-average distribution of surface temperature, cloud cover, radiative cloud forcing and so on is remarkably good bearing in mind the various approximations and simplifications which are necessary in any climate model built at this minimal level of detail. On the other hand, the simulation is not perfect, and indeed could probably be improved considerably by a more rigorous tuning of certain of the input parameters. For instance, all of the globally constant long-wave parameters such as ε_a, ε_c, $F^{abt}{}_G$ (see Table 1 section b) have been assigned values to the nearest 0.05, and each of them could legitimately be tuned within at least this range – and indeed could be set to have values varying with latitude and longitude.

Even with more careful tuning, it is doubtful whether model simulations on their own can resolve (for instance) the question as to whether case A or case B is the more appropriate representation of the real physical situation. The bottom line is that a number of issues need to be addressed before embarking on a detailed tuning exercise.

There is for instance the issue of whether the two basic constraints of the model can in practice be combined into one grand maximization of the overall entropy production associated with the turbulent energy flows of the planet. There is still an issue as to whether the *internal* radiative exchanges (as between ground and cloud for instance) should be included in the overall maximized entropy production as well as the turbulent energy flows – this because those particular radiative energy transfers involve turbulent energy transfer in a sort of 'series connection' from the point of the radiation absorption in the atmosphere to the level of the effective atmospheric temperature. There is an issue as to whether any of the input parameters might be calculated (rather than specified) as part of the maximization process. And there is a related (and perhaps ultimate) question as to the limit of the detail of the Earth's actual climate distribution that might simulated using only the MEP principle.

7. ACNOWLEDGEMENTS

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TABLE 1. INPUT PARAMETER VALUES

Input parameters used to define the MEP model. The latitudes are the (effective) mid-latitudes of the ten equal-area latitude zones of each hemisphere. The solar input I is the TOA annual mean calculated from the seasonal values of Manabe and Moller (1991). The values of g_o (effectively the albedo of a pure Rayleigh-scattering atmosphere on its own) and d_o (the albedo of the cloudy-sky portion of the atmosphere on its own) were roughly interpolated from information in Paltridge and Platt, 1976, pages 132 and 103) taking into account the effective average zenith angle (ZA) of the sun for each zone (Hartman, 1994). The surface albedos α and emissivities ε are zonal averages from the geographic distributions used in the 2-D model. The values of the other parameters are estimates roughly tuned to within 0.05, except that the values of F^{ct}_{cb}, z_0 and k_o have been tuned rather more carefully to give a reasonable overall output. The parameter k_o is the clear-sky atmospheric absorption over the oceans at all latitudes. The clear-sky absorption over land is calculated as a function of surface albedo according to the parameterisation of equation 7.

(a) LATITUDE DEPENDENT PARAMETERS

Latitude (^{0})	ZA (^{0})	I (W/m^{-2})	g_o	d_o	α (S.Hem)	α (N.Hem)	F^{ct}_{cb}	ε (S.Hem)	ε (N.Hem)
72.0	77	186	0.130	0.57	0.300	0.250	0.80	0.99	0.99
58.5	66	242	0.095	0.43	0.073	0.098	0.80	0.99	0.98
48.7	59	288	0.080	0.39	0.062	0.104	0.80	0.99	0.98
40.6	54	324	0.070	0.37	0.061	0.093	0.80	0.99	0.98
33.4	49	355	0.060	0.35	0.067	0.096	0.78	0.99	0.97
26.7	46	376	0.055	0.34	0.078	0.108	0.75	0.98	0.97
20.4	43	393	0.050	0.33	0.083	0.098	0.71	0.98	0.97
14.4	41	406	0.047	0.32	0.075	0.083	0.70	0.99	0.99
8.6	39	413	0.045	0.31	0.074	0.079	0.70	0.99	0.99
2.8	38	420	0.045	0.30	0.072	0.071	0.70	0.99	0.99

(b) PARAMETERS ASSUMED INDEPENDENT OF LATITUDE

Variable	Both cases	Case A	Case B
F^{abt}_G	0.55		
F^{cb}_G	0.85		
k_c	0.20		
ε_a	0.75		
ε_c	1.00		
z_0		1.07	1.09
k_o		0.19	0.18

TABLE 2. SENSITIVITY TO INPUT PARAMETERS.

Model-calculated partial derivative sensitivities of global average cloud cover θ, surface temperature T and the surface-to-atmosphere sensible and latent heat flux HLE for case A and case B. The global averages of the parameters at the point where the partial derivatves were evaluated were: $\theta = 0.50$, $T = 289.4$ K and $HLE = 126.2$ Wm^{-2} for case A; $\theta = 0.62$, $T = 287.2$ K and $HLE = 124.9$ Wm^{-2} for case B. The value of Δp for L corresponds to a 1% increase in L, and for all other parameters corresponds to an increase of their value (which in all cases is a unit-less fraction) by 1.0 so that the sensitivities are in units appropriate to a partial derivative. (The actual change imposed on these unit-less parameters for calculation of the sensitivities was 0.01).

	Case A			Case B		
Parameter	$\Delta\theta/\Delta p$	$\Delta T/\Delta p$	$\Delta HLE/\Delta p$	$\Delta\theta/\Delta p$	$\Delta T/\Delta p$	$\Delta HLE/\Delta p$
L	0	0.72	1.3	0	0.71	1.2
α	0.64	-60.1	-164.3	0.3	-52.5	-162.7
g_o	6.1	-139.2	-125.0	5.8	-130.7	-134.8
d_o	-7.8	95.1	-90.4	-7.8	88.3	-74.3
k	9.4	-152.8	-158.4	9.8	-168.4	-185.6
k_c	-9.6	155.7	-114.6	-10.6	179.5	-80.9
ε	-0.65	-5.7	-86.1	-0.96	3.4	-79.6
ε_a	-9.3	182.7	244.5	-8.4	162.0	231.9
$F^{abt}{}_G$	-4.5	22.9	65.0	-5.1	44.0	82.5
$F^{cb}{}_G$	3.1	-95.3	73.6	4.8	-126.3	39.6
$F^{ct}{}_{cb}$	0.81	-56.8	27.0	1.9	-82.6	8.8
z_0	-0.07	1.9	6.5	-0.05	1.4	3.4
X_0/X	–	–	–	-0.004	-0.03	-4.0

APPENDIX A

The mathematics of finding the maximum of HLE/T reduces to using the energy balance eqs (14) and (15) of the main text so as to find the temperature (defined by η_{max}) and cloud cover θ_{max} at which

$$\frac{\partial}{\partial T}\left(\frac{HLE}{T}\right)=0 \quad \Leftrightarrow \quad 4\eta.\frac{\partial HLE}{\partial \eta}-HLE=0 \qquad \text{.........(A1)}$$
$$\Leftrightarrow \quad c_3\eta^3+c_2\eta^2+c_1\eta+c_0=0$$

with:

$$c_3 = 3D(SC - RD)$$
$$c_2 = ADSL + DS\Delta X - 7SLCB + LQCD - LPD^2 - \Delta X_0 D^2 + 6DLBR$$
$$c_1 = 3L^2QBC - 5LQD\Delta X - 5L^2QAD + 3SAL^2B + 3SLB\Delta X + 2DL^2PB + 2\Delta X_O DLB - 3L^2B^2R$$
$$c_0 = L^2B(LQA + Q\Delta X - BPL - B\Delta X_O$$

That is, we need to solve

$$\Pi(\eta) = 0 \quad \text{for the condition when} \quad d\Pi/d\eta < 0 \qquad \text{............(A2)}$$

where Π is of the standard form of a polynomial of the third degree, namely

$$\Pi = \eta^3 + p\eta^2 + q\eta + r = 0 \qquad \text{.........(A3)}$$

$$p = c_2/c_3, \quad q = c_1/c_3, \quad r = c_0/c_3$$

and where c_1, c_2 and c_3 are real numbers. We can define:

$$a = (3q - p^2)/3$$
$$b = (2p^3 - 9pq + 27r) / 27$$
$$Q = b^2/4 + a^3/27$$

In our case Q is negative and there are three distinct roots. Only one of them, η_{max}, satisfies the condition of eq. (A1). It can be expressed as

$$\eta_{max} = -\rho^{1/3} . [\cos(\kappa/3) - 3^{1/2}\sin(\kappa/3)] - p/3 \qquad \text{............ (A4)}$$

where $\rho = \{(-b/2)^2 - Q\}^{1/2}$ and $\kappa = \arctan\{(-Q)^{1/2}/(-b/2)\}$

Substitution of η_{max} back in the energy balance eq. (14) in the main text gives the expression for cloud cover

$$\theta_{max} = (\eta_{max}.C - A.L - \Delta X) / (\eta_{max}.D - L.B) \qquad \text{.........(A5)}$$

Equations (A4) and (A5) can then be used to replace the original eqs (17) and (18) from O'Brien and Stephens.

CAPTIONS TO FIGURES

Figure 1: Schematic diagram of the energy fluxes into and out of a latitude zone.

Figure 2: Contour plot of the distribution of surface albedo (percent) used in the 2-D model.

Figure 3: Schematic diagram of the cloudy-sky and clear-sky solar fluxes into and out of a model grid box.

Figure 4: Schematic diagram of the long-wave fluxes out of the atmosphere of a model grid box.

Figure 5: Zonal average distributions of meridional energy flux X, surface temperature T, cloud cover θ and vertical heat flux HLE obtained from case A of the 2-D model. The dashed curves are observed values.

Figure 6: Zonal average distributions of meridional energy flux X, surface temperature T, cloud cover θ and vertical heat flux HLE obtained from case B of the 2-D model. The dashed curves are observed values.

Figure 7: Distribution of fractional cloud cover for the case A version of the 2-D model.

Figure 8: Distribution of fractional cloud cover for the case B version of the 2-D model.

Figure 9: Distributions of zonal-average cloud forcing (solar, long-wave and net forcing) at the top of the atmosphere from the case A version of the 2- D model. The sign convention is positive inwards, so that the solar and net forcing is negative, and the long-wave forcing is positive. The left diagram is of the pure partial derivative of radiation flux with respect to fractional cloud cover. The right diagram is of those derivatives multiplied by the existing cloud cover at the relevant latitude.